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HEAT TRANSFER TO PLANE NON-NEWTONIAN COUETTE FLOW

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Abstract—Heat transfer to laminar non-Newtonian Couette flow with pressure gradient is investigated in the present work. The general velocity distribution for a power-law non-Newtonian fluid is developed. The heat transfer model, which includes the viscous dissipation, is numerically simulated by using an implicit finite difference method. Comparison of the present numerical solutions for a special case with the previous ones shows very good agreement for two different types of boundary conditions. The effects of several dimensionless parameters, such as reciprocal of dimensionless pressure gradient, pseudoplastic index and viscous dissipation parameter, on the heat transfer characteristics are numerically explored.

NOMENCLATURE

- a_i, integration constant;
- A, constant, $-a_1/m\alpha$;
- C_p, constant pressure heat capacity;
- k, thermal conductivity;
- m, consistency index;
- n, pseudoplastic index;
- Nu, Nusselt number, $h\delta/k$;
- P, pressure;
- T, temperature;
- T_0, inlet temperature;
- T_1, wall temperature of the bottom plate;
- T_2, wall temperature of the top plate;
- T_m, bulk temperature;
- U, dimensionless velocity;
- v_x, local velocity;
- V, velocity of the moving top plate;
- x, axial coordinate;
- X, dimensionless axial coordinate, $xk/\rho C_p V\delta^2$;
- y, coordinate perpendicular to the flow;
- y_m, location from the bottom where the maximum velocity occurs;
- Y, dimensionless coordinate perpendicular to the flow, y/δ;
- Y_m, dimensionless location from the bottom plate where the maximum velocity occurs, y_m/δ.

Greek symbols

- α, parameter in equation (5), $\left(-\dfrac{\delta}{m}\dfrac{dP}{dx}\right)$;
- β, dimensionless parameter, $V(n+1)/n\delta\alpha^{1/n}$;
- δ, distance between the two plates;
- η, dimensionless temperature, $(T_1-T_0)/(T_2-T_0)$;
- θ, dimensionless temperature, $(T-T_0)/(T_2-T_0)$;
- θ_m, dimensionless bulk temperature, $(T_m-T_0)/(T_2-T_0)$;
- σ, dimensionless parameter, $\left(\dfrac{n+1}{n}\right)^{n+1} mV^{n+1}/k\delta^{n-1}\beta^{n+1}(T_2-T_0)$;
- τ, shear stress.

1. INTRODUCTION

LAMINAR heat transfer to liquids in plane Couette flow is a problem of practical interest and has been receiving an increasing amount of attention in the past several years. Thermal sterilization of liquid foods and biological materials and heat transfer in the bearing-journal devices are typical examples of its applications [1–4].

This problem has been investigated by several researchers. Important literature regarding the past investigations can be found in [1–4]. A study of the previous investigations indicates that most of these works were confined to heat transfer to Newtonian Couette flow only. The corresponding case in non-Newtonian flow has received very little attention thus far. The main reason for considering the plane non-Newtonian Couette heat transfer is that a large number of liquid foods, fermentation broths and lubricating oils exhibit non-Newtonian rheological behavior [5–7]. Investigation of this problem therefore may provide more relevant information of the heat transfer characteristics across the flow passage.

The only available literature to date dealing with heat transfer in non-Newtonian Couette flow was that of Tien [8] who extended the Schlichting's approach for simple Newtonian heat transfer in a Couette flow to the non-Newtonian case. The problem investigated by Tien [8] involved a number of restrictive assumptions which need further elaborations. For example, the axial heat convection was neglected and no axial pressure gradient was considered in his work. The first assumption was particularly difficult to justify for this forced convection problem. Because of these assumptions, the problem he treated was oversimplified. Although an analytical solution was explicitly obtained, it would not be able to show the true heat transfer characteristics of the non-Newtonian Couette flow. In order to improve Tien's solution, these restrictive assumptions have to be released and this is the purpose of the present work.

2. VELOCITY DISTRIBUTION OF THE NON-NEWTONIAN COUETTE FLOW

The velocity distribution for the Newtonian Couette flow with pressure gradient can be represented by [1–4]:

$$v_x = \frac{y}{\delta}V - \frac{\delta^2}{2\mu}\frac{dP}{dx}\left(\frac{y}{\delta}\right)\left(1-\frac{y}{\delta}\right). \quad (1)$$

If the pressure gradient is neglected in the above equation, it reduces to a linear function of position across the flow passage, which applies to Newtonian as well as non-Newtonian Couette flows.

A simple momentum balance of flow yields:

$$\frac{d\tau}{dy} = -\frac{dP}{dx}, \quad (2)$$

which is readily integrated to give

$$\tau = \left(-\frac{dP}{dx}\right)y + a_1. \quad (3)$$

The non-Newtonian fluid is assumed to be characterized by the power-law model which for the present case is represented by

$$\tau = -m\left(\frac{dv_x}{dy}\right)^n \quad (4)$$

because the velocity gradient is positive for the entire flow. Combining equations (3) and (4) leads to:

$$\frac{dv_x}{dy} = \alpha^{1/n}(A-Y)^{1/n}, \quad (5)$$

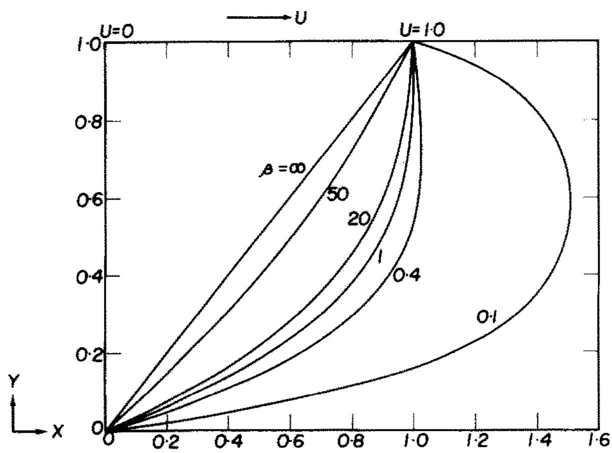

FIG. 1. The non-Newtonian velocity distributions with $n = 0.4$.

An interesting point exhibited by the above equation is that if the dimensionless group, $\delta^2/2\mu$ $(-dP/dx)$, is equal to or less than one, the maximum fluid velocity occurs at the top plate. On the other hand, if this dimensionless pressure group is greater than one, the maximum fluid velocity takes place between the two plates. This characteristic is retained also for the non-Newtonian case, as shown in Fig. 1. Such a flow characteristic renders the derivation of velocity distribution for the non-Newtonian Couette flow far more complicated than for the Newtonian case. It may be for this reason that the velocity distribution for the non-Newtonian Couette flow is still not available in the open literature. Hence, it must be derived first before the model formulation of the present problem can be undertaken. Due to the flow characteristic mentioned above, separate expressions of velocity distribution are needed for the dimensionless pressure group less and greater than one, respectively. In the following section, the velocity distribution for the former case will be developed first.

in which the pressure gradient is included in α and A contains the integration constant a_1 and α. The exact forms of α and A are given in the Nomenclature. Integration of equation (5) yields

$$v_x = -\alpha^{1/n}\delta\left(\frac{n}{n+1}\right)(A-Y)^{(n+1)/n} + a_2, \quad (6)$$

which is applicable to the entire flow. Using the boundary condition $v_x = V$ at $Y = 1$ and $v_x = 0$ at $Y = 0$ and eliminating a_2 leads to:

$$v_x = \alpha^{1/n}\delta\left(\frac{n}{n+1}\right)[A^{(n+1)/n} - (A-Y)^{(n+1)/n}] \quad (7)$$

$$V = \alpha^{1/n}\delta\left(\frac{n}{n+1}\right)[A^{(n+1)/n} - (A-1)^{(n+1)/n}]. \quad (8)$$

If the reciprocal of the dimensionless pressure gradient for the present non-Newtonian case is defined as

$$\beta = \frac{n+1}{n}\,\frac{V}{\delta^{(n+1)/n}\left(-\dfrac{1}{m}\dfrac{dP}{dx}\right)^{1/n}}. \quad (9)$$

Equations (7) and (8) then can be rearranged to yield:

$$\frac{v_x}{V} = \frac{1}{\beta}[A^{(n+1)/n} - (A-Y)^{(n+1)/n}] \quad (10)$$

$$A^{(n+1)/n} - (A-1)^{(n+1)/n} = \beta. \quad (11)$$

It can easily be shown that, as $n = 1$, equations (10) and (11) reduce to equation (1). Equation (11) serves to determine A in terms of n and β. Knowing A, the velocity distribution then can be constructed from equation (10).

defined as before. Using the boundary condition $v_x = V$ at $Y = 1$ to eliminate a_3 yields:

$$v_x = V + \alpha^{1/n}\delta\left(\frac{n}{n+1}\right) \times [(1-A)^{(n+1)/n} - (Y-A)^{(n+1)/n}], \quad (15)$$

which can be rewritten, using equation (9), as

$$\frac{v_x}{V} = 1 + \frac{1}{\beta}[(1-A)^{(n+1)/n} - (Y-A)^{(n+1)/n}], \quad \text{for} \quad 1 \geqslant Y \geqslant Y_m. \quad (16)$$

Table 1. Numerical values of A in terms of β and n

β \ n	0.2	0.4	0.6	0.8	1.0
0.1	0.682451	0.570554	0.559372	0.552823	0.550000
0.2	0.764832	0.652729	0.617841	0.605462	0.600000
0.3	0.818206	0.716965	0.674627	0.657738	0.650000
0.4	0.858377	0.772728	0.729153	0.709477	0.700000
0.5	0.890899	0.821461	0.781067	0.760514	0.750000
0.6	0.918386	0.864577	0.830211	0.810697	0.800000
0.7	0.942287	0.903217	0.876573	0.859882	0.850000
0.8	0.963492	0.938254	0.920235	0.907930	0.900000
0.9	0.982596	0.970347	0.961329	0.954699	0.950000
1.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.5	1.069913	1.122964	1.166662	1.207748	1.250000
2.0	1.122462	1.219881	1.307219	1.398253	1.500000
2.5	1.164995	1.301493	1.432336	1.578148	1.750000
3.0	1.200943	1.372848	1.546772	1.750368	2.000000
5.0	1.307697	1.598670	1.938873	2.388679	3.000000
10.0	1.468056	1.980568	2.697504	3.794659	5.500000
30.0	1.764688	2.835158	4.766083	8.441074	15.50000
50.0	1.923328	3.375477	6.300192	12.45084	25.50000

It must be borne in mind that equations (10) and (11) apply only for $\beta \geqslant 1$ which has a maximum fluid velocity at the top plate. For the case $\beta \leqslant 1$, the maximum fluid velocity occurs somewhere between the two plates, say at Y_m from the bottom plate. In fact, equation (7) is still applicable to the present case in the region $Y_m \geqslant Y \geqslant 0$ because of similar flow characteristic, but A is no longer given by equation (11). In the region $1 \geqslant Y \geqslant Y_m$, however, a separate expression is needed. In this region, equation (4) becomes:

$$\tau = m\left(-\frac{dv_x}{dy}\right)^n, \quad (12)$$

because of negative velocity gradient. Invoking the same derivations as above, the following equation can be obtained from equations (3) and (12)

$$\frac{dv_x}{dy} = -\alpha^{1/n}(Y-A)^{1/n}, \quad (13)$$

which is integrated to

$$v_x = -\alpha^{1/n}\delta\left(\frac{n}{n+1}\right)(Y-A)^{(n+1)/n} + a_3, \quad (14)$$

where the dimensionless parameters are similarly

Remembering that

$$\frac{v_x}{V} = \frac{1}{\beta}[A^{(n+1)/n} - (A-Y)^{(n+1)/n}], \quad \text{for} \quad Y_m \geqslant Y \geqslant 0. \quad (17)$$

As mentioned before, A in equations (16) and (17) is different from that in the previous case with $\beta \geqslant 1$. To determine A for the present case, the condition that the velocities from equations (16) and (17) must be equal at $Y = Y_m$ can be used. It is also noted that at $Y = Y_m$, the velocity gradient disappears. Therefore from equation (13), it is apparent that $Y_m = A$. By equating equation (16) to equation (17) at $Y = A$, there yields

$$A^{(n+1)/n} - (1-A)^{(n+1)/n} = \beta. \quad (18)$$

With the value of A determined from equations (11) and (18), the velocity distribution for the power-law non-Newtonian fluids can be fully established from equation (10) for $\beta \geqslant 1$ and equations (16) and (17) for $\beta \leqslant 1$. The values of A from the two transcendental equations were determined by the Newton–Raphson method [9] for different n and β and are listed in Table 1.

3. THE HEAT TRANSFER MODEL

The steady state heat transfer equation with constant physical properties can be written as:

$$C_p \rho v_x \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} - \tau\left(\frac{\mathrm{d}v_x}{\mathrm{d}y}\right). \tag{19}$$

The second term in the right hand side represents the viscous dissipation which was also considered by El-Ariny and Aziz [4] and Tien [8]. The viscous dissipation term is especially important for the highly viscous non-Newtonian fluids.

In terms of the dimensionless variables and parameters, equations (19) can be rewritten as

$$U(Y)\frac{\partial \theta}{\partial X} = \frac{\partial^2 \theta}{\partial Y^2} + \sigma f(Y), \tag{20}$$

where the dimensionless velocity distribution $U(Y)$ is given by equation (10) for $\beta \geqslant 1$ and equations (16) and (17) for $\beta \leqslant 1$, and σ is a dimensionless parameter which reduces to the product of the Eckert and Prandtl number for the Newtonian case. The viscous dissipation function, $f(Y)$, is given by

$$f(Y) = (A - Y)^{(n+1)/n}, \quad \text{for } \beta \geqslant 1 \text{ and } 1 \geqslant Y \geqslant 0, \tag{21}$$

and

$$f(Y) = (A - Y)^{(n+1)/n}, \quad \text{for } \beta \leqslant 1 \text{ and } A \geqslant Y \geqslant 0 \tag{22}$$

$$= (Y - A)^{(n+1)/n}, \quad \text{for } \beta \leqslant 1 \text{ amd } 1 \geqslant Y \geqslant A. \tag{23}$$

Two different sets of boundary conditions are considered here. They are:

$$X = 0; \quad \theta = 0 \tag{24}$$

$$Y = 0; \quad \theta = \eta \tag{25}$$

$$Y = 1; \quad \theta = 1, \tag{26}$$

for boundary condition A, and

$$X = 0; \quad \theta = 0 \tag{27}$$

$$Y = 0; \quad \theta = 1 \tag{28}$$

$$Y = 1; \quad \partial\theta/\partial Y = 0, \tag{29}$$

for boundary condition B. The Nusselt number is evaluated as

$$Nu = -\frac{\delta}{T_2 - T_m}\left(\frac{\partial T}{\partial y}\bigg|_{y=0 \text{ or } \delta}\right) = -\frac{1}{1-\theta_m}\left(\frac{\partial \theta}{\partial Y}\bigg|_{Y=0 \text{ or } 1}\right), \tag{30}$$

where the dimensionless bulk temperature is given by

$$\theta_m = \frac{\int_0^1 \theta U(Y)\mathrm{d}Y}{\int_0^1 U(Y)\mathrm{d}Y}. \tag{31}$$

The previous investigators [1–4] solved the corresponding equations for the Newtonian case by a semi-analytic method which requires accurate evaluation of the eigenvalues. In the present work, this method was not attempted and instead the implicit Crank–Nicolson finite difference method [9] was used. This method is stable and accurate.

4. DISCUSSION AND RESULTS

In order to test the accuracy of the implicit finite difference scheme, several runs were made for the Newtonian Couette heat transfer as a special case so that comparison can be made with the previous results. The present numerical solutions are essentially identical to the semi-analytical solutions of El-Ariny and Aziz [4]. They [4] have pointed out that insufficient number of eigenvalues used in the infinite series of semi-analytic solution can lead to underestimation of both the dimensionless temperature and the Nusselt number. The difficulty in accurate evaluation of the eigenvalues, however, can be avoided by using the finite difference method.

Other typical results for Case A are shown in Figs. 2–5. Figure 2 demonstrates the effect of the dimensionless pressure gradient group β on the mean dimensionless temperature. The mean dimensionless temperature increases quite significantly with increasing β. According to the definition, β is the reciprocal of the pressure gradient. Low β corresponds to a high pressure gradient and hence the flow becomes fast as β decreases, as seen in Fig. 1. Fast flow reduces the residence time of the fluid inside the channel and thus a lower mean dimensionless temperature is expected. It is also observed in this figure that the mean dimensionless temperature seems to be rather sensitive to β when β is less than one. For β greater than one, its effect becomes less

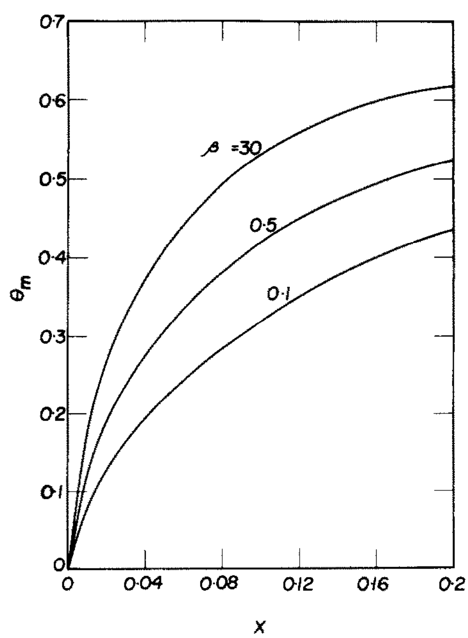

FIG. 2. The effect of reciprocal of dimensionless pressure gradient on the dimensionless mean temperature for Case A with $n = 0.4$, $\sigma = 0$, and $\eta = 0$.

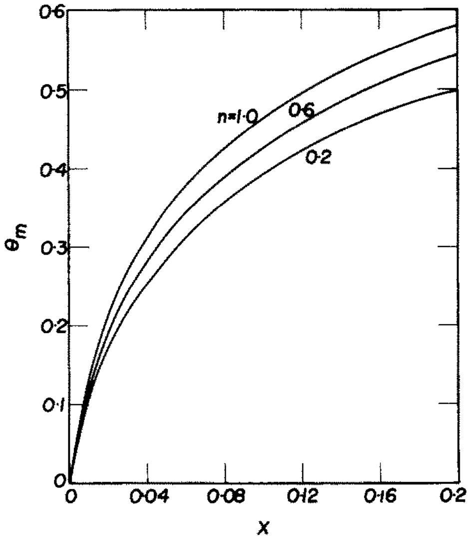

FIG. 3. The effect of the pseudoplastic index on the dimensionless mean temperature for Case A with $\beta = 0$, $\sigma = 0$ and $\eta = 0$.

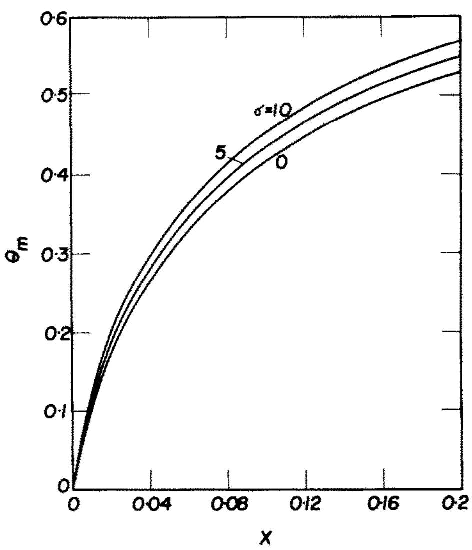

FIG. 4. The effect of the viscous dissipation parameter on the dimensionless mean temperature for Case A with $\beta = 0.5$, $n = 0.4$ and $\eta = 0$.

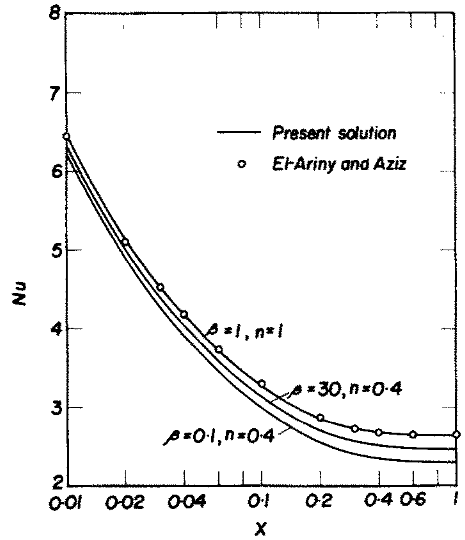

FIG. 5. The local Nusselt number vs the axial position for Case A with $\sigma = 0$ and $\eta = 0$.

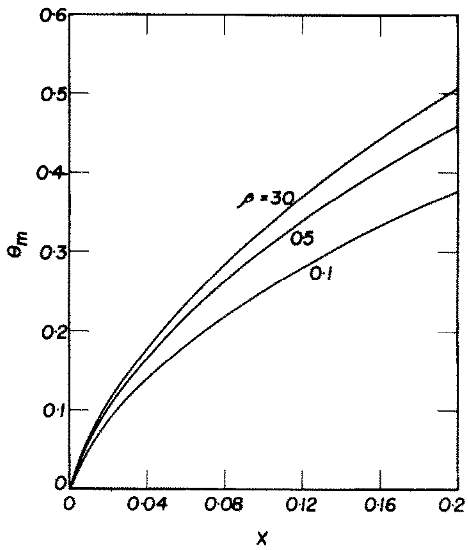

FIG. 6. The effect of the reciprocal of dimensionless pressure gradient on the dimensionless mean temperature for Case B with $n = 0.4$ and $\sigma = 0$.

appreciable. This may be due to the fact that the shape of the velocity profile, as shown in Fig. 1, changes more significantly in the region with β less than one.

The effect of the pseudoplastic index n on the dimensionless mean temperature is shown in Fig. 3. The value of n ranging from 0.2 to 1.0 covers a large number of non-Newtonian liquid foods, polymer melts and lubricant oils [5–7]. According to equation (10), as the value of n increases to infinity, the velocity profile becomes a linear function of Y. For a smaller n, a nonlinear velocity profile appears and deviation from the linear profile increases with decreasing n. This implies that the residence time of the fluid increases with increasing n. This explains for the increasing dimensionless mean temperature at a larger n.

As mentioned previously, the viscous dissipation can be rather significant in many practical circumstances because of high fluid viscosity of many non-Newtonian fluids. The effect of viscous dissipation on the dimensionless mean temperature is displayed in Fig. 6. It is obvious that the viscous dissipation tends to increase the dimensionless mean temperature as anticipated. This is due to irreversible conversion of mechanical energy to thermal energy.

The local Nusselt numbers for different values of β are given in Fig. 5. Also included in this figure is a special case with $n = 1$ so that comparison with that of El-Ariny and Aziz [4] can be made. The agreement between the present solution and the semi-analytic one of El-Ariny and Aziz [4] appears to be very good. The local Nusselt numbers for all the cases tend to asymptotically approach some

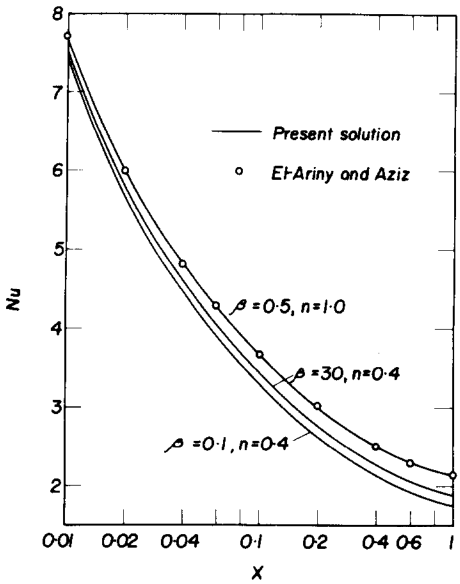

FIG. 7. The local Nusselt number vs the axial position for Case B with $\sigma = 0$.

constants as the dimensionless axial distance becomes sufficiently large because of full development of the thermal field.

The numerical results for Case B are shown in Figs. 6 and 7. It is clear that the general heat transfer characteristics of the previous case are still retained for the present one. Comparison of Figs. 6 and 2 indicates that the temperature development of the fluid is faster for Case A than the present one. Fast thermal development results in smaller temperature gradient which may be mainly responsible for the lower Nusselt number for Case A especially for $X \leqslant 0.1$.

5. CONCLUSIONS

An analytical procedure is presented in this study for calculating the thermal development in a non-Newtonian Couette flow with pressure gradient. The general heat transfer model is simulated, using an implicit finite difference method, to investigate the effects of several dimensionless parameters on the heat transfer characteristics. The present numerical solution for a special case is in good agreement with the previous one obtained semi-analytically. Because of the accuracy and stability of the finite difference method, relevant information can be generated without difficulty for the purpose of design of heat transfer equipment.

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TRANSFERT THERMIQUE EN PAROI POUR UN ECOULEMENT DE COUETTE DE FLUIDE NON-NEWTONIEN

Résumé—On étudie le transfert thermique pour un écoulement laminaire de fluide non-newtonien selon Couette, avec gradient de pression. La distribution des vitesses est établie pour une loi puissance de fluide non-newtonien. Le modèle qui inclut la dissipation visqueuse est simulé numériquement en utilisant une méthode implicite aux différence finies. Une comparaison des résultats numériques, dans un cas particulier, avec d'autres antérieurs montre un très bon accord pour deux types différents de conditions aux limites. On analyse les effets de plusieurs paramètres adimensionnels tels que l'homologue du gradient de pression, l'indice pseudoplastique et le paramètre de dissipation visqueuse.

WÄRMEÜBERGANG IN EINER EBENEN NICHT-NEWTONSCHEN COUETTE-STRÖMUNG

Zusammenfassung—In der vorliegenden Arbeit wird der Wärmeübergang in einer laminaren nicht-newtonschen Couette-Strömung mit Druckgradienten untersucht. Die allgemeine nach einem Potenzgesetz verlaufende Geschwindigkeitsverteilung wird für ein nicht-newtonsches Fluid entwickelt. Das Wärmetransport-Modell, welches die zähigkeitsbedingte Dissipation beinhaltet, wird unter Anwendung eines finiten Differenzenverfahrens numerisch simuliert. Der Vergleich für einen bestimmten Fall der erhaltenen numerischen Lösung mit früheren Lösung zeigt für zwei verschiedene Arten von Randbedingungen gute Übereinstimmung. Die Einflüsse von einigen dimensionslosen Parametern, wie dem Reziprokwert des dimensionslosen Druckgradienten, dem pseudoplastischen Index und dem Parameter der viskosen Dissipation auf das Wärmeübergangsverhalten werden numerisch untersucht.

ТЕПЛОПЕРЕНОС К ОБОБЩЕННОМУ КУЭТТОВСКОМУ ТЕЧЕНИЮ НЕНЬЮТОНОВСКОЙ ЖИДКОСТИ

Аннотация — Получено общее распределение скоростей для неньютоновской «степенной» жидкости. Система уравнений для переноса тепла, включающая вязкую диссипацию, решается численно с помощью неявного конечно-разностного метода. Сравнение результатов численных решений выбранного конкретного случая с данными, полученными ранее, дает хорошее совпадение для двух различных типов граничных условий. Численно исследовано влияние на характеристики переноса тепла нескольких безразмерных параметров, таких как обратная величина безразмерного градиента давления, индекс псевдопластичности и параметр вязкой диссипации.